

Abstract Algebra Problem Bank

– updated in Fall 2022, managed by Donghi Lee and Yeongrak Kim

1. (1) Show that the dihedral group $D_3 = \langle a, b \mid a^3 = 1, b^2 = 1, ab = ba^{-1} \rangle$ is solvable, but not nilpotent.

(2) Prove that the dihedral group $D_n = \langle a, b \mid a^n = 1, b^2 = 1, ab = ba^{-1} \rangle$ is nilpotent if and only if $n = 2^k$.

(Hint: For the proof of (\Rightarrow) , write $n = 2^k m$ with $2 \nmid m$, and show that $m = 1$. You may consider the subgroup $P = \langle a^m, b \rangle$ of D_n , and may use Burnside-Wilandt Theorem and the equality $ba^2 = a^{-1}ba$ in D_n .)

2. Let G be a finite nilpotent group with $|G| = n$. Show that G satisfies the converse of Lagrange's Theorem, that is, for every positive integer d such that $d \mid n$, there exists a subgroup of order d .

(Hint: You may use Burnside-Wilandt Theorem and Sylow's First Theorem.)

3. (1) Let G be a group. If $A \trianglelefteq B \leq G$ and $H \leq G$, prove that $(B \cap H)/(A \cap H)$ is isomorphic to a subgroup of B/A .

(Hint: You may use the First Isomorphism Theorem.)

(2) Using (1), show that any subgroup of a polycyclic group is also polycyclic.

(Definition: A group G is called polycyclic if it has a subnormal series with cyclic factors.)

4. Express the dihedral group D_n as a semidirect product of two subgroups.

5. Let G be a group presented by $\langle b, c, t \mid c^9 = 1, b^{-1}cb = c^{-1}, t^{-1}bt = bc^3, t^{-1}ct = c \rangle$.

(1) Let F be a free group with basis $\{b, c, t\}$. Show that there exists a unique group homomorphism $\bar{\psi} : F \rightarrow G$ such that

$$\bar{\psi} : b \mapsto b, \quad c \mapsto c^3, \quad t \mapsto t^3.$$

(2) Show that there exists a unique group homomorphism $\tilde{\psi} : G \rightarrow G$ such that

$$\tilde{\psi} : b \mapsto b, \quad c \mapsto c^3, \quad t \mapsto t^3.$$

6. Prove that every group G of order p^2 (p prime) is abelian.

(Hint: You may use Class Equation to show that $Z(G) \neq \{1\}$.)

7. Let R be a ring with identity 1_R , and let A be an ideal of R . Prove that A is contained in some maximal ideal of R .

(Hint: You may use Zorn's Lemma.)

8. Show that the ring R consisting of all rational numbers with denominators not divisible by some (fixed) prime p is a local ring.

(Hint: You may use a localization of \mathbb{Z} .)

9. (1) Is the polynomial $x^2 + 1$ irreducible in $\mathbb{Z}[x]$? Give your reason.

(2) Is the polynomial $x^2 + 1$ irreducible in $\mathbb{Z}[[x]]$? Give your reason.

10. (1) If I, J are ideals of a ring R , prove that $I + J = \{i + j \mid i \in I, j \in J\}$ is an ideal of R .

(2) Let F be a field. Prove that the ideal $\langle x \rangle + \langle y \rangle$ of $F[x, y]$ is not a principal ideal.

11. If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is a split exact sequence of R -modules, prove that $B \cong A \oplus C$ as R -modules.

12. Let m and n be relatively prime integers greater than 1, and let $R = \mathbb{Z}_{mn}$. Prove that \mathbb{Z}_m and \mathbb{Z}_n are projective R -modules, but are not free.

13. Let R and S be rings and let ${}_R A, B_S, {}_R C_S$ be (bi)modules as indicated. Prove that there is an isomorphism α of abelian groups

$$\alpha : \text{Hom}_R(A, \text{Hom}_S(B, C)) \rightarrow \text{Hom}_S(B, \text{Hom}_R(A, C))$$

such that $\alpha(f)(b)(a) = f(a)(b)$ for all $f \in \text{Hom}_R(A, \text{Hom}_S(B, C))$, $a \in A$ and $b \in B$.

14. Let F be a field. Prove that every F -module is both projective and injective.

15. For every positive integer n , show that \mathbb{Z}^n is not an injective \mathbb{Z} -module. (Hint: You may use Baer's Criterion.)

16. Prove that \mathbb{Q} is a flat \mathbb{Z} -module.

17. Let A and B be R -modules over a commutative ring R . Prove that $A \otimes_R B \cong B \otimes_R A$ as R -modules.

18. For arbitrary positive integers m and n , compute $\text{Hom}_{\mathbb{Z}}(m\mathbb{Z}, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n))$. (Hint: You may use Adjoint Associativity.)

19. If R is a PID, prove that every submodule of a projective R -module is projective.

20. Show that complex conjugation from \mathbb{C} to itself is a homomorphism of \mathbb{R} -algebras, but not a homomorphism of \mathbb{C} -algebras.

21. Let $G = GL(2, \mathbb{R})$ be the set of 2×2 invertible matrices over \mathbb{R} . We define a binary operation $*$: $G \times G \rightarrow G$ as

$$A * B := \frac{AB + BA}{2}$$

where AB and BA are given by the usual matrix multiplication. Determine whether $(G, *)$ is an abelian group or not.

22. Show that any finite subgroup of \mathbb{Q}/\mathbb{Z} is cyclic.

23. Show that the symmetry group of a regular tetrahedron (= the group of rotations and reflections which fixes the given tetrahedron) is isomorphic to S_4 .

24. Let $V = \mathbb{R}^4$ be a 4-dimensional \mathbb{R} -vector space, and let $G = GL(4, \mathbb{R})$ be the group of 4×4 invertible matrices over \mathbb{R} . Let

$$S := \{W \subset V \mid W \text{ is a 2-dimensional subspace of } V\}$$

be the set of all 2-dimensional subspaces in V . We identify a vector $v = (v_1, v_2, v_3, v_4) \in$

V , $v_i \in \mathbb{R}$ with a column vector $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$.

(1) Let $A \in G$ and let $W \in S$. Show that $A(W) := \{Aw \mid w \in W\}$ is also an element in S , where Aw is given by the usual matrix multiplication.

(2) Show that the map $(A, W) \mapsto A(W)$ defines a G -action on S .

(3) Let $W_1 \in S$ be the subspace spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$. What is the orbit of

W_1 under this G -action? What is the stabilizer subgroup G_{W_1} ?

25. How many Sylow 3-groups and Sylow 5-groups are in the alternating group A_5 ?

26. Determine whether the following rings are UFD or not, and give your reason.

(1) $(\mathbb{Z}/5\mathbb{Z})[x]$

(2) $\mathbb{Z}[\sqrt{-5}][x]$

(3) $\mathbb{Z}[x, y]$

27. Let R be a commutative ring with unity 1, and let $a_1, \dots, a_r \in R$ be elements such that $\langle a_1, \dots, a_r \rangle = \langle 1 \rangle$. Show that $\langle a_1^{n_1}, \dots, a_r^{n_r} \rangle = \langle 1 \rangle$ for any positive integers n_1, \dots, n_r .

28.

(1) Let $c \in \mathbb{C}$ be a complex number. Show that the ideal $I = \langle x - c \rangle$ in $\mathbb{C}[x]$ is maximal.

(2) Show that any maximal ideal I in $\mathbb{C}[x]$ is of the form $I = \langle x - c \rangle$ for some $c \in \mathbb{C}$.

(3) Find a maximal ideal J in $\mathbb{R}[x]$ which is not of the form $\langle x - c \rangle$ for some $c \in \mathbb{R}$.

(4) Let $c_1, \dots, c_n \in \mathbb{C}$ be complex numbers. Show that $\langle x_1 - c_1, x_2 - c_2, \dots, x_n - c_n \rangle$ is a maximal ideal in $\mathbb{C}[x_1, \dots, x_n]$.

29. Let k be a field, and let V_1, \dots, V_n be k -vector spaces. Suppose that there is an exact sequence

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow \dots \rightarrow V_n \rightarrow 0.$$

Show that $\sum_{i=1}^n (-1)^i \dim_k V_i = 0$.

30. Is there a finite field F which is algebraically closed? Explain why.

31. Determine the extension degree $[E : \mathbb{Q}]$ of the splitting field E of each of the following polynomials over \mathbb{Q} .

(1) $x^4 - 1$

(2) $x^3 - 2$

(3) $x^4 + 1$

(4) $x^3 + x^2 - 2x - 1$

32. Let $f = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$.

(1) Show that f is irreducible.

(2) Check that the equation $f(x) = 0$ has 4 distinct roots in \mathbb{R} , namely,

$$\alpha_1 = \sqrt{2 + \sqrt{2}}, \alpha_2 = -\sqrt{2 + \sqrt{2}}, \alpha_3 = \sqrt{2 - \sqrt{2}}, \alpha_4 = -\sqrt{2 - \sqrt{2}}.$$

(3) Let E be the splitting field of f over \mathbb{Q} , and let $G = \text{Gal}(E/\mathbb{Q})$. Let $\sigma \in G$ be a \mathbb{Q} -automorphism of E such that $\sigma(\alpha_1) = \alpha_3$. Show that $\sigma(\alpha_3) = \alpha_2$ (**Hint**: what are $\sigma(\alpha_1^2)$ and $\sigma(\alpha_1\alpha_3)$?).

(4) Conclude that $G \simeq \mathbb{Z}/4\mathbb{Z}$.

(5) Find all the intermediate fields between E and \mathbb{Q} .

33. Let $\zeta_9 = e^{2\pi i/9}$ be a primitive 9th root of unity.

(1) Find the minimal polynomial of ζ_9 over \mathbb{Q} .

(2) What is the Galois group $Gal(\mathbb{Q}(\zeta_9)/\mathbb{Q})$?

(3) Describe all the intermediate fields between $\mathbb{Q}(\zeta_9)$ and \mathbb{Q} .